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A NOTE ON AN EXTENSION OF RATIONAL BOUNDS FOR THE t-TAIL AREA TO ARBITRARY DEGREES OF FREEDOM

Andrew P. Soms

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

The bounds of Soms (1977) for the tail area of the t-distribution with integral degrees of freedom are extended to arbitrary positive degrees of freedom. Three numerical examples are provided.

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SIGNIFICANCE AND EXPLANATION

Real degrees of freedom arise in the modification of the two-sample t-test when the variances cannot be assumed to be equal. For small degrees of freedom both linear interpolation and standard computer routines may be unsatisfactory. The present paper provides a simple technique for satisfactory estimates of tail probabilities.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

A NOTE ON AN EXTENSION OF RATIONAL BOUNDS FOR THE t-TAIL AREA TO ARBITRARY DEGREES OF FREEDOM

Andrew P. Soms

1. Introduction and Statement of Extension

In applications, such as the Welch modification of the two-sample t-test when variances cannot be assumed to be equal, it is sometimes desired to evaluate the t-tail area for nonintegral degrees of freedom. It is therefore of practical interest to extend the results of Soms (1977) to arbitrary real degrees of freedom. While one could use the relationship between the t and beta distributions indicated in Section 3, this would consume substantially more computer time because of the numerical integration and would introduce errors if either limit of integration was close to 0 or 1. We introduce notation and state the result in this Section, indicate the proofs in 2, and give some numerical examples in 3.

For arbitrary real $k>0$, let

$$f_k(t) = c_k(1+t^2/k)^{-(k+1)/2}, \quad c_k = \frac{\Gamma((k+1)/2)}{\Gamma(k/2)(\pi k)^{1/2}},$$

$$\bar{F}_k(x) = 1-F_k(x) = \int_x^\infty f_k(t)dt,$$

and

$$R_x = \bar{F}_k(x)/[(1+x^2/k)f_k(x)].$$

Also, for $k>2$, let $\gamma_{max} = 4c_k^2/(1-4c_k^2)$ and $\gamma_{min} = \dfrac{k}{2(k+2)c_k^2} - 1$, and for $k<2$, interchange the definitions of γ_{max} and γ_{min}. Then we have the following extension of Theorems 2.1 and 2.2 of Soms (1977).

<u>Theorem 1.1</u>: Let $p(x,\gamma) = \dfrac{1+\gamma}{(x^2+4c_k^2(1+\gamma)^2)^{1/2}+\gamma x}$. Then

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$$p(x,\gamma_{min}) < R_x < p(x,\gamma_{max}) \text{ ,} \tag{1.1}$$

or, equivalently,

$$(1+\frac{x^2}{k})f_k(x)p(x,\gamma_{min}) < \overline{F}_k(x) < (1+\frac{x^2}{k})f_k(x)p(x,\gamma_{max}). \tag{1.2}$$

This is a generalization of Soms (1977) where (1.1), or equivalently (1.2), was shown to hold for integral k. It was also shown there that if k=2, $\gamma_{max} = \gamma_{min} = \gamma_2$ and $R_x = p(x,\gamma_2)$, and hence this case need not be considered further.

2. An Outline of the Proofs

We first note that the asymptotic expansion given in Soms (1976) for the tail area of the t-distribution is valid for arbitrary k>0 even though it was stated for k a positive integer, because this fact was not used in the derivation. Hence the heuristic derivation of (1.1) remains exactly as before. To insure that the proof of Theorem 1.1 given for the integral case works for k real, all that needs to be verified is that the c_k satisfy the lemmas in the Appendix and the relationships of Theorems 2.1 and 2.2 of Soms (1977). We state these lemmas and relationships and indicate their proofs.

Lemma 2.1: For real k>0, the c_k form an increasing sequence with limit $1/(2\pi)^{1/2}$.

Proof: That the limit is $1/(2\pi)^{1/2}$ is shown by using Stirling's formula. To obtain the result it suffices to show that $\frac{d \ln c_k}{dk} = g(k) > 0$ for all k>0. By differentiation,

$$g(k) = \frac{1}{2}\,\Psi(\frac{k+1}{2}) - \frac{1}{2}\,\Psi(\frac{k}{2}) - \frac{1}{2k} \text{ ,}$$

where $\Psi(x)$ is the digamma function (see, e.g., Abramowitz and Stegun, 1965, p. 258). From Artin (1964, p. 17),

$$h(k) = \frac{1}{2}\,\Psi\left(\frac{k+1}{2}\right) - \frac{1}{2}\,\Psi\left(\frac{k}{2}\right) = \sum_{i=0}^{\infty} \frac{1}{(k+2i)(k+2i+1)} .$$

Let $f(x) = ((k+2x)(k+2x+1))^{-1}$. Then it can be shown by differentiation that f is convex and by considering the area under f(x) from 0 to n and letting $n \to \infty$ it follows that

$$\int_0^{\infty} f(x)dx + \frac{1}{2}\,f(0) \le h(k) ,$$

or

$$\frac{1}{2}\,\ell n\,\frac{k+1}{k} + \frac{1}{2}\,\frac{1}{k(k+1)} \le h(k) ,$$

since $\int \frac{1}{(k+2t)(k+2t+1)}\,dt = \frac{1}{2}\,\ell n\,\frac{k+2t}{k+2t+1}$. The above is a special case of the Euler-Maclaurin summation formula and was shown to the author by Harris (1978). Therefore it suffices to show that

$$\ell n\,\frac{k+1}{k} + \frac{1}{k(k+1)} > \frac{1}{k} ,$$

or, since $\ell n\,\frac{k+1}{k} = \ell n(k+1) - \ell n\,k > \frac{1}{k+1}$, that

$$\frac{1}{k+1} + \frac{1}{k(k+1)} \ge \frac{1}{k} ,$$

which is true for all $k>0$, concluding the proof.

__Lemma 2.2__: For $k>0$, $1-4c_k^2 > 0$.

__Proof__: Follows from Lemma 2.1.

__Lemma 2.3__: For $k>2$, $8c_k^2 - 1 > 0$ and for $k<2$, $8c_k^2 - 1 < 0$.

__Proof__: Follows from Lemma 2.1 and the observation that $8c_2^2 - 1 = 0$.

__Lemma 2.4__: For $k>2$, $c_k^2 < k/(6k+4)$ and for $k<2$, $c_k^2 > k/(6k+4)$.

__Proof__: Let $h(k) = \ell n(k/6k+4))^{\frac{1}{2}} - \ell n\,c_k$. Then $\lim_{k \to 0} h(k) = 0$, $h(2) = 0$ and it must be shown that $h(k) < 0$, $0 < k < 2$ and $h(k) > 0$, $k>2$. By differentiation and the recursion formula for the Ψ function (see, e.g. Abramowitz and Stegun, p. 258),

$\Psi(x+1) = \Psi(x) + 1/x$,

$$h'(k) = \frac{3k+1}{2(3k+2)(k+1)} - \frac{1}{2}\left(\Psi\left(\frac{k+1}{2}+1\right) - \Psi\left(\frac{k}{2}+1\right)\right) .$$

By differentiation, it follows that for $k \leq (2^{\frac{1}{2}}-1)/3 \cong .138$, $f(k) = \frac{3k+1}{2(3k+2)(k+1)}$ is an increasing function and for $k \geq .138$ decreasing. Also, from Artin (1964, p. 17) it follows that $g(k) = \frac{1}{2}\left(\Psi\left(\frac{k+1}{2}+1\right) - \Psi\left(\frac{k}{2}+1\right)\right)$ is both a positive and decreasing function of k. Hence, for k such that $k_L \leq k \leq k_R \leq .138$,

$$h'(k) < f(k_R) - g(k_R) , \tag{2.1}$$

and for k such that $.138 \leq k_L \leq k \leq k_R$,

$$h'(k) > f(k_R) - g(k_L) . \tag{2.2}$$

Also, since c_k and $d_k = (k/(6k+4))^{\frac{1}{2}}$ are both increasing functions of k, for k such that $k_L \leq k \leq k_R$,

$$d_{k_L} - c_{k_R} \leq d_k - c_k \leq d_{k_R} - c_{k_L} . \tag{2.3}$$

Using (2.1), (2.2), and (2.3), it follows that $h'(k) < 0$, .0-.1(.05), $h(k) < 0$, .1-.3(.01), .3-1.1(.05), and $h'(k) > 0$, 1.1-14.2(.1). From Lemma 2.1, $h(k) > 0$ if $(k/(6k+4))^{\frac{1}{2}} > (1/2\pi)^{\frac{1}{2}}$ or $k \geq 14.2$. Combining the above results gives the conclusion.

To complete the extension it is necessary to prove the three following lemmas used in Soms (1977).

<u>Lemma 2.5</u>: For $k>2$, $c_k^2 < k/(4k+8)$, and for $k<2$, $c_k^2 > k/(4k+8)$.

<u>Proof</u>: For $k>2$, $k/(4k+8) > k/(6k+4)$ and for $k<2$, $k/(4k+8) <$ $k/(6k+4)$, and so using Lemma 2.4, the result follows.

<u>Lemma 2.6</u>: For $k>2$, $16kc_k^4 + (4k+4)c_k^2 > k$ and for $k<2$, $16kc_k^4 +$ $(4k+4)c_k^2 < k$.

<u>Proof</u>: By some algebra, Lemma 2.6 is equivalent to

$c_k^2 \gtrless \left[2(\frac{k+1}{k})\left[(1+(\frac{2k}{k+1})^2)^{1/2}+1\right]\right]^{-1}$ for $k \gtrless 2$. By using the recursion formula for the Γ function this is further equivalent to

$$h_1(k) = \frac{2}{k+1}\frac{\Gamma^2(\frac{k+1}{2}+1)}{\Gamma^2(\frac{k}{2}+1)\pi} \gtrless h_2(k) = \left[(1+(\frac{2k}{k+1})^2)^{1/2}+1\right]^{-1} \text{ for } k \gtrless 2.$$

Let $g(k) = \ell n\, h_1(k) - \ell n\, h_2(k)$. Then $\lim_{k\to 0} g(k) = g(2) = 0$ and we must show that $g(k) < 0$, $0 < k < 2$, and $g(k) > 0$, $k > 2$. By differentiation,

$$g'(k) = -\frac{1}{k+1} + \left[\Psi(\frac{k+1}{2}+1) - \Psi(\frac{k}{2}+1)\right] + \frac{4k}{(k+1)^3}\times$$

$$\left[(1+(\frac{2k}{k+1})^2)^{1/2}\left((1+(\frac{2k}{k+1})^2)^{1/2}+1\right)\right]^{-1} =$$

$$-\frac{1}{k+1} + g_1(k) + g_2(k) .$$

Now $g'(k) < -\frac{1}{k+1} + g_1(k) + 2k$ and for k such that $k_L \le k \le k_R$, $g'(k) < -\frac{1}{k_R+1} + g_1(k_L) + 2k_R$. Also, since for $k \ge 1$, $g_2(k)$ is a decreasing function of k, we have $g'(k) > -\frac{1}{k_L+1} + g_1(k_R) + g_2(k_R)$ for $1 \le k_L \le k \le k_R$. Hence $g'(k) < 0$, .0-.15(.01), $g(k) < 0$, .15-.25(.001) and .25-1.60(.01), $g'(k) > 0$, 1.-3.4(.1), $g(k) > 0$, 3.4-5.0(.01) and 5.0-17(.1) and since $16k\, c_k^4 + (4k+4)c_k^2 > 16k\, c_k^4 + 4k\, c_k^2 = k(16c_k^4 + 4c_k^2)$ and $16c_{17}^4 + 4c_{17}^2 > 1$, $g(k) > 0$, $k \ge 17$. Combining the above yields the result.

<u>Lemma 2.7</u>: For $k>2$, $c_k^2 > \frac{(k)(k+4)}{8(k+1)(k+2)}$ and for $k<2$, $c_k^2 < \frac{(k)(k+4)}{8(k+1)(k+2)}$.

<u>Proof</u>: By using the recursion formula for the Γ function, it is equivalent to show

$$h_1(k) = \frac{\Gamma^2(\frac{k+1}{2}+1)}{\Gamma^2(\frac{k}{2}+1)} \gtrless h_2(k) = \frac{(k+1)(k+4)}{8(k+2)}, \quad k \gtrless 2.$$

Let $g(k) = \ell n\, h_1(k) - \ell n\, h_2(k)$. Then $\lim_{k\to 0} g(k) = 0$ and $g(2) = 0$.

By differentiation, $g'(k) = \Psi(\frac{k+1}{2}+1) - \Psi(\frac{k}{2}+1) + \frac{1}{k+2} - (\frac{1}{k+1} + \frac{1}{k+4}) = g_1(k) - g_2(k)$. Since g_1 and g_2 are decreasing functions of k and $a(k) = (k)(k+4)/(8(k+1)(k+2))$ is increasing for $k \leq (12)^{\frac{1}{2}} + 2 \doteq 5.46$ we have for k such that $k_L \leq k \leq k_R$,

$$c_k^2 - a(k) \leq c_{k_R}^2 - a(k_L), \qquad k_R \leq 5.46 \tag{2.4}$$

and

$$g_1(k_R) - g_2(k_L) \leq g'(k) \leq g_1(k_L) - g_2(k_R).$$

Using (2.4), we have $g'(k) < 0$, .0-.45(.05), $g(k) < 0$, .45-1.15(.01), $g'(k) > 0$, 1.15-9.05(.01) and $g(k) > 0$ for $k > 9$ since $c_k^2 > \frac{(k)(k+4)}{8(k+1)(k+2)}$ if $c_k^2 > \frac{k+4}{8(k+2)}$ and the latter is true for $k = 9$. Combining the above concludes the proof.

All the calculations involved in the proofs were done on a computer using Γ and Ψ routines.

3. Some Numerical Examples

Suppose $k = .2$ and $x = 4$. Then using Theorem 1.1, we find that $c_{.2} = .19748$, $\gamma_{max} = .16557$, $p(4,\gamma_{max}) = .24859$, $\gamma_{min} = .18482$, $p(4,\gamma_{min}) = .24857$ and $.28468 < \overline{F}_{.2}(4) < .28471$. The value given by a computer routine using the relationship $\overline{F}_k(t) = .5 + .5\, I_x(1/2,k/2)$, $x = t^2/k/(1+t^2/k)$, is .28471. If $k = 2.5$ and $x = 2$, then $c_{2.5} = .361809$, $\gamma_{max} = 1.099178$, $p(2,\gamma_{max}) = .445704$, $\gamma_{min} = 1.121967$, $p(2,\gamma_{min}) = .445287$ and $.07868 < \overline{F}_{2.5}(2) <$ .0787[illegible]. The computer value is .07870. Finally, if $k = .1$, $x = 25$, then $c_{.1} = .14809$, $\gamma_{max} = .085642$, $p(25,\gamma_{max}) = .039997$, $\gamma_{min} = .096161$, $p(25,\gamma_{min}) = .039997$ and $.30251 < \overline{F}_{.1}(25) < .30251$. The computer value is .30251. The above statements of course reflect rounding.

These numerical examples indicate that Theorem 1.1 would be well suited to a small sample Monte Carlo study of the Welch t-test.

References

Abramowitz, Milton and Stegun, Irene A. (1965), Handbook of Mathematical Functions, Washington: National Bureau of Standards.

Artin, Emil (1964), The Gamma Function, New York: Holt, Rinehart and Winston.

Harris, Bernard (1978), Private Communication.

Soms, Andrew P. (1976), "An Asymptotic Expansion for the Tail Area of the t-Distribution," Journal of the American Statistical Association, 71, 728-30.

Soms, Andrew P. (1977), "Rational Bounds for the t-Tail Area with an Application to Bonferroni t-Percentiles," Technical Summary Report No. 1807, Mathematics Research Center, University of Wisconsin-Madison, and to appear in the Journal of the American Statistical Association.

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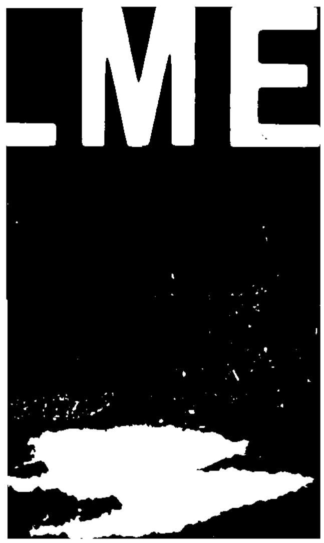
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